Exercise 7 Stochastic Models of Manufacturing Systems 4T400, 9 June

1. In a dairy barn, cows arrive at the milking robots (which are located in parallel) according to a Poisson stream with a rate of 20 cows per hour. The cows are milked by the robots in order of arrival. Data (in minutes) on the milking times is available.

   (a) What is the minimum number of robots required?

   (b) For this minimum number of robots, estimate the utilization of the robots, the mean waiting time of the cows, the mean number waiting and the fraction of cows that has to wait longer than 5 minutes. Compare these results to the ones with one additional robot.

   (c) Typically cows in a herd are socially ranked; high-ranked cows may push away the low-ranking ones while waiting for the robots. Assume that 10% of the cows are dominant. Estimate the mean waiting time of the high-ranking cows and the low-ranking cows (in case the minimum number of robots is being used).

   Answer:

   (a) The mean milking time is $E(B) = 8.41$ minutes, with a standard deviation of 2.52 minutes, and the arrival rate is $\lambda = \frac{1}{3}$ cows per minute. Hence the minimum number of required robots is 3 (since $\lambda \frac{E(B)}{c} < 1$ for $c = 3$, and it is > 1 for $c = 2$).

   (b) For $c = 3$ robots, we have $\rho = \lambda \frac{E(B)}{c} = 0.93$, and

   $$E(W) \approx \frac{\Pi_W}{1-\rho} \frac{E(R)}{c} = 20.4 \text{ (min)}, \quad E(L^q) \approx 6.81,$$

   where

   $$E(R) = \frac{E(B)}{2}(1 + c_B^2) = 4.58 \text{ (min)}, \quad \Pi_W \approx \Pi_W(M/M/c) = 0.88.$$

   Further,

   $$P(W > t) \approx \Pi_W e^{-c(1-\rho)t/E(R)}, \quad t \geq 0,$$

   so

   $$P(W > 5) \approx 0.71.$$

   For $c = 4$ we get

   $$\rho = 0.70, \quad E(W) \approx 1.65 \text{ (min)}, \quad E(L^q) \approx 0.55, \quad P(W > 5) \approx 0.12.$$

   (c) Let λ_1 be the arrival rate of high-ranking cows, and λ_2 the arrival rate of low-ranking cows. Further, ρ_1 is the utilization due to high-ranking cows and W_1 is the waiting time of high-ranking cows. Similar notation is used for high-ranking cows. Then

   $$E(W_1) \approx \frac{\Pi_W}{1-\rho_1} \frac{E(R)}{c} = 1.48 \text{ (min)}, \quad E(W_2) \approx \frac{\Pi_W}{(1-\rho_1)(1-\rho_1-\rho_2)} \frac{E(R)}{c} = 22.55 \text{ (min)}.$$

   So high-ranking cows strongly benefit, whereas the low-ranking ones suffer just a bit.

2. Consider again the dairy barn, where cows arrive at the robots according to a Poisson process with a rate of 20 cows per hour. After having visited the robots, the cows want to have some concentrate food. Data (in minutes) on the feeding times at the concentrate feeder are also available.

(a) What is the minimum number of feeders required?

(b) Estimate the mean numbers and the mean total flow time (at the robots and feeders), in case the minimum numbers of robots and feeders is being used. Compare these estimates to the ones with one additional robot and feeder.

(c) Now suppose that only 70% of the cows want some concentrate food after being milked; the other 30% leave without having concentrate food. What is now the minimum number of feeders required? For this minimum number, estimate again the mean number and the mean total time spent at the concentrate feeders for those cows who do want some concentrate.

Answer:

(a) The mean feeder time is $E(B) = 6.37$ minutes, with a standard deviation of 6.25 minutes, and the arrival rate is $\lambda = \frac{1}{3}$ cows per minute. Hence the minimum number of required feeders is 3 (since $\lambda \frac{E(B)}{c} < 1$ for $c = 3$, and it is > 1 for $c = 2$).

(b) The mean number and mean flow time at the robots has already been calculated for $c = 3, 4$ in the previous assignment. For $c = 3$ robots, we get as estimate for the squared coefficient of variation of the inter-departure times,

$$c_D^2 \approx 1 + (1 - \rho^2)(c_A^2 - 1) + \frac{\rho^2}{\sqrt{c}}(c_B^2 - 1) = 0.54.$$

where c_A^2 is the squared coefficient of variation of the inter-arrival times, which is 1 in case of Poisson arrivals. Hence, for $c = 3$ feeders, we have $\rho = \lambda \frac{E(B)}{c} = 0.71$, and

$$E(W) \approx \frac{\Pi_W}{1-\rho} \frac{E(B)}{2c}(c_D^2 + c_B^2) = 2.75 \text{ (min)}, \quad E(L^q) \approx 0.92,$$

where

$$\Pi_W \approx \Pi_W(M/M/c) = 0.50.$$

Similarly, for $c = 4$ robots, we get

$$c_D^2 \approx 1 + (1 - \rho^2)(c_A^2 - 1) + \frac{\rho^2}{\sqrt{c}}(c_B^2 - 1) = 0.78,$$

and for $c = 4$ feeders, we have $\rho = \lambda \frac{E(B)}{c} = 0.53$, and

$$E(W) \approx \frac{\Pi_W}{1-\rho} \frac{E(B)}{2c}(c_D^2 + c_B^2) = 0.60 \text{ (min)}, \quad E(L^q) \approx 0.20,$$

where

$$\Pi_W \approx \Pi_W(M/M/c) = 0.21.$$

(c) Now the arrival rate at the feeders is $\lambda = \frac{7}{30}$, so the minimum number of required feeders is 2. For 3 robots, we have $c_D^2 \approx 0.54$, and the squared coefficient of variation of the inter-arrival times at the feeders then becomes

$$c_A^2 \approx 0.7 c_D^2 + 0.3 = 0.68.$$

So for $c = 2$ feeders, we have $\rho = 0.74$ and the mean waiting time at the feeders is approximately

$$E(W) \approx \frac{\Pi_W}{1-\rho} \frac{E(B)}{2c}(c_A^2 + c_B^2) = 6.3 \text{ (min)}, \quad E(L^q) \approx 1.47,$$

where

$$\Pi_W \approx \Pi_W(M/M/c) = 0.63.$$

3. Consider two workstation in series. Each station has a single machine. The mean processing time is 1 hour and the coefficient of variation is 3, on both machines. Jobs arrive at a rate of 0.9 jobs per hour, and they come from all over the plant, so it is reasonable to assume that they arrive according a Poisson stream.

   (a) Estimate the mean flow time of jobs.

   (b) Now suppose a *flexible machine* is available with the same capacity but with less variability, say with a coefficient of variation equal to $\frac{1}{2}$. At which workstation should we replace the existing machine by the flexible one to get the largest reduction in mean flow time? How big is this reduction?

   Answer:

   (a) An approximation of the mean flow time at workstation i is

   $$E(S_i) = \frac{\rho_i}{1-\rho_i}\,\frac{1}{2}\,E(B_i)(c_{A_i}^2 + c_{B_i}^2) + E(B_i), \quad i = 1, 2,$$

   where $\rho_i = 0.9$, $E(B_i) = 1$, $c_{B_i}^2 = 9$, $c_{A_1}^2 = 1$ (Poisson) and

   $$c_{A_2}^2 \approx \rho_1^2 c_{B_1}^2 + (1-\rho_1^2)c_{A_1}^2 = 7.48.$$

   Hence,

   $$E(S_1) = 46 \text{ (hours)}, \quad E(S_2) \approx 75.16 \text{ (hours)}, \quad E(S) = E(S_1)+E(S_2) = 121.16 \text{ (hours)}.$$

   (b) Replace the first machine, then $c_{B_1}^2 = \frac{1}{4}$ and $c_{A_2}^2 = c_{D_1}^2 \approx 0.3925$, so

   $$E(S_1) = 6.625 \text{ (hours)}, \quad E(S_2) \approx 43.27 \text{ (hours)}, \quad E(S) = E(S_1)+E(S_2) = 49.9 \text{ (hours)}.$$

   By replacing the second one, we get

   $$E(S_1) = 46 \text{ (hours)}, \quad E(S_2) \approx 34.94 \text{ (hours)}, \quad E(S) = E(S_1)+E(S_2) = 80.94 \text{ (hours)}.$$

   So it is better to replace the first machine, which leads to a reduction in mean flow time of more than 70 hours.

4. The table below gives the mean processing time $\frac{1}{\mu}$ (in hours), coefficient of variation c, and cost (in Keuro) for tools sets in an assembly line. The desired mean flow time is 16 hours. The predicted (Poisson) demand is 0.5 part per hour. Find the least-cost configuration meeting the mean flow time target.

Station	Tool set 1			Tool set 2		
	$\frac{1}{\mu}$	c	cost	$\frac{1}{\mu}$	c	cost
1	1	2	50	1	1.8	85
2	1.7	0.1	50	1.5	0.7	145
3	1.2	1	100	1.4	0.5	120
4	1.1	0.75	20	0.6	1.3	75

   Answer:

   (a) An approximation of the mean flow time at workstation i is

   $$E(S_i) = \frac{\rho_i}{1-\rho_i}\,\frac{1}{2}\,E(B_i)(c_{A_i}^2 + c_{B_i}^2) + E(B_i), \quad i = 1, 2, 3, 4,$$

   where $E(B_i) = \frac{1}{\mu_i}$, $c_{A_1} = 1$ (Poisson) and $c_{A_i} = c_{D_{i-1}}$ for $i = 2, 3, 4$. The least-cost configuration, for which the mean total flow time $E(S_1) + \cdots + E(S_4)$ is less than the target of 16 hours, is tool set 2 for stations 1 and 4 and tool sets 1 for stations 2 and 3 (with a mean flow time of approximately 15.8 hours).

5. A production line of 3 machines in series is producing parts. Parts arrive according to a Poisson stream with a rate of λ parts per hour. The processing times are exponential, with a mean of 5 minutes at machine 1, 4 minutes at machine 2 and 4.5 minutes at machine 3. Each machine has ample buffer space. After machine 3, parts are inspected (which takes negligible time). 10% of the parts is rejected and has to go through the production process again.

   (a) What is the bottleneck machine?

   (b) What is maximum arrival rate λ such that the system is stable?

   (c) Assume $\lambda = 10$ parts per hour. Calculate the mean flow time of parts.

   (d) What is the probability that there are more than 10 parts waiting at machine 1 for processing?

   (e) Suppose that the production process can be improved, such that 8%, instead of 10%, will be rejected. Calculate the reduction in mean flow due to this (minor) improvement.

   Answer:

   (a) The bottleneck machine is the slowest one, so machine 1.

   (b) Let λ_i denote the total inflow at machine i. Then

   $$\lambda_1 = \lambda + 0.1\lambda_3, \quad \lambda_3 = \lambda_2 = \lambda_1.$$

   So

   $$\lambda_i = \frac{10}{9}\,\lambda.$$

   The utilization of machine i is $\rho_i = \lambda_i \frac{1}{\mu_i}$, so (where λ is parts per *minute*)

   $$\rho_1 = \lambda\frac{50}{9}, \quad \rho_2 = \lambda\frac{40}{9}, \quad \rho_1 = \lambda\frac{45}{9}.$$

   Hence, the maximum arrival rate is $\frac{9}{50}$ parts per minute, thus $10\frac{4}{5}$ parts per hour.

   (c) For $\lambda = \frac{1}{6}$ parts per minute, we get

   $$E(S_1) = \frac{\frac{1}{\mu_1}}{1-\rho_1} = 67.5 \text{ (min)}, \quad E(S_2) = 15.4 \text{ (min)}, \quad E(S_3) = 27 \text{ (min)},$$

   and thus the total mean flow time is 109.9 (min).

   (d) The probability of n or more parts at station 1 is ρ_1^n. Hence, the probability of more than 10 parts waiting at machine 1 is $\rho_1^{12} = (25/27)^{12} = 0.4$.

   (e) Then we get

   $$\lambda_i = \frac{10}{9.2}\,\lambda = 0.181.$$

   So $E(S) = E(S_1) + E(S_2) + E(S_3) = 52.6 + 14.5 + 24.3 = 91.4$ (min), which is a reduction of nearly 20%!

6. A manufacturing system consisting of 3 workstations (labeled 1, 2, 3), each with a single machine, is simultaneously processing 4 types of jobs. These jobs arrive from all over the factory, so it is reasonable to assume Poisson inflow. Processing times are exponential. The arrival rates, routings and mean processing times (per visit and job type) are listed in the table below.

Job type	Arrival rate (jobs/hour)	Routing	Mean processing time (hour)
1	0.1	1,2,3	1,1,2
2	0.1	2,1,2	3,1,2
3	0.1	3,1,3,1	4,4,2,1
4	0.15	3,2,1	1,2,1

(a) Determine the bottle neck workstation.
(b) Determine the number of jobs at each workstation.
(c) Determine the mean total flow time for each job type.

Answer:

(a) The arrival rate at station i is

$$\lambda_1 = 0.1 + 0.1 + 2 \cdot 0.1 + 0.15 = 0.55, \quad \lambda_2 = 0.45, \quad \lambda_3 = 0.45.$$

The mean (and second moment of the) processing time of an arbitrary job at station 1 is

$$E(B_1) = \frac{2}{11} \cdot (1+1+4+1) + \frac{3}{11} \cdot 1 = \frac{17}{11}, \quad E(B_1^2) = \frac{4}{11} \cdot (1+1+16+1) + \frac{6}{11} \cdot 1 = \frac{82}{11},$$

where we used that $E(X^2) = \frac{2}{\mu^2}$ if X is exponential with rate μ. Similarly,

$$E(B_2) = 2, \quad E(B_2^2) = \frac{80}{9}, \quad E(B_3) = \frac{19}{9}, \quad E(B_3^2) = \frac{102}{9},$$

and also

$$E(R_1) = \frac{E(B_1^2)}{2E(B_1)} = \frac{82}{34} = 2.41, \quad E(R_2) = 2.22, \quad E(R_3) = 2.68.$$

So $\rho_1 = 0.85$, $\rho_2 = 0.9$ and $\rho_3 = 0.95$, and hence, station 3 is the bottle neck.

(b) The mean number in station i (waiting plus in process) can be estimated by

$$E(L_i) = \lambda_i \frac{\rho_i}{1 - \rho_i} E(R_i) + \rho_i,$$

yielding

$$E(L_1) = 8.4, \quad E(L_2) = 9.9, \quad E(L_3) = 23.9.$$

(c) The mean waiting time of an arbitrary job is station i is

$$E(W_1) = \frac{\rho_1}{1 - \rho_1} E(R_1) = 13.7, \quad E(W_2) = 20, \quad E(W_3) = 50.9.$$

The mean flow time of type 1 jobs is

$$E(S_1) = E(W_1) + 1 + E(W_2) + 1 + E(W_3) + 2 = 88.6,$$

and similarly,

$$E(S_2) = 59.7, \quad E(S_3) = 140.2, \quad E(S_4) = 88.6 \text{ (hour)}.$$